First Name:____________________ **Last Name:**____________________

OSU Number:__________________ **Signature:**________________________

OKLAHOMA STATE UNIVERSITY
Department of Mathematics

MATH 4143/5043 (Advanced Calculus)
Instructor: Dr. Mathias Schulze

MIDTERM 2
October 30, 2009

Duration: 50 minutes

No aids allowed.

This examination paper consists of **3** pages and **3** questions. Please bring any discrepancy to the attention of an invigilator. The number in brackets at the start of each question is the number of points the question is worth.

Answer 3 questions.

To obtain credit, you must give arguments to support your answers.

For graders' use:

	Score
1 (12)	
2 (12)	
3 (12)	
Total (36)	

1. [12] Formulate definitions (describe the data involved and list the defining properties). Pick 3 out of 5 subproblems.
   (a) What is a critical point?
   (b) Define the directional drivative and the total derivative of a function.
   (c) What is a function of class C^k?
   (d) Define the Hessian matrix.
   (e) What is an antiderivative?

   Solution: See textbook and lecture notes.

2. [12] Formulate theorems (list all hypotheses and formulate the statement). Pick 3 out of 5 subproblems.
   (a) Mean Value Theorem (differential or integral version, 1 variable)
   (b) Local Approximation Theorem (n variables)
   (c) Chain Rule (n variables)
   (d) Taylor's Formula (n variables)
   (e) 2nd Derivative Test (n variables)

   Solution: See textbook and lecture notes.

3. [12] Prove statements (give rigorous arguments based on the definitions). Pick 3 out of 5 subproblems.
   (a) If $f\colon [a,b] \to \mathbb{R}$ is continuous and $\int_a^x f = \int_x^b f$ for all $x \in (a,b)$, then $f = 0$.
   (b) If $f\colon [a,b] \to \mathbb{R}$ is continuous, $F(x) = \int_a^x f(t)dt$ is an antiderivative for f on (a,b).
   (c) Integral Cauchy-Schwarz Inequality: For $f, g\colon [a,b] \to \mathbb{R}$ continuous,

   $$\left(\int_a^b fg\right)^2 \leq \left(\int_a^b f^2\right) \cdot \left(\int_a^b g^2\right).$$

   (d) The subset of points in $[a,b] \times [c,d]$ with coordinates in $\mathbb{Q}$ does not have an area.
   (e) The integral of a bounded function over a finite domain is zero.

   Solution:

   (a) Let F be an antiderivative of f on (a,b). Then, by the fundamental theorem of calculus, the given equality becomes $2F(x) = F(b) - F(a)$. This shows that F is constant and hence $f = F' = 0$ on (a,b), and hence also on $[a,b]$ by continuity of f.

(b) See proof of Theorem 3.4.5.[1]

(c) By monotony and linearity of the integral we have

$$0 \le \int_a^b (f+\lambda g)^2 = \int_a^b f^2 + 2\lambda \int_a^b fg + \lambda^2 \int_a^b g^2.$$

This means that the right hand side polynomial of λ has no two different real zeros (in which case it would have negative values). Then the discriminant $\Delta = \left(\int_a^b fg\right)^2 - \int_a^b f^2 \int_a^b g^2$ must be ≤ 0 and the claim follows.

(d) Let Q the set of points in $R = [a,b]\times[c,d]$ with coordinates in $\mathbb{Q}$, and let N be any grid on R. Recall that $\mathring{Q} = \emptyset$ and $\bar{Q} = R$. So $\underline{S}_N(Q) = 0$ and $\overline{S}_N(Q) = A(R)$, for all grids N on R, by definition of these Riemannian sums. It follows that $\underline{A}(Q) = 0 < A(R) = \overline{A}(Q)$ and hence Q has no area.

(e) Consider $f\colon D \to \mathbb{R}$ with $D \subset \mathbb{R}^n$ finite, and pick a rectangle R containing D. Consider

$$F = \begin{cases} f(x), & x \in D, \\ 0, & x \in R \setminus D. \end{cases}$$

Then by definition, $\int_D f = \int_R F$. But if $\#D = k$ then, for any grid N on R, both $\overline{S}_N(F)$ and $\underline{S}_N(F)$ involve at most k summands. Setting $M = \max|f(D)|$ [2] and $\delta = \delta(N)$, this shows that

$$-4kM\delta^2 \le \underline{S}_N(F) \le \overline{S}_N(F) \le 4kM\delta^2.$$

It follows that $\int_R F$ exists and equals zero.

End of examination
Total pages: 3
Total marks: 36

[1] I noticed here that there is a problem in the textbook: Having an antiderivative makes sense only in an open interval, because this is where derivatives in general have been defined. So the statement of Theorem 3.4.5 should be reformulated as "f has an antiderivative on (a,b)". However Theorem 3.4.7 remains true, but the proof requires more work.

[2] Note that we do not need the assumtion that f is bounded. This is automatic by finiteness of D. I was hoping that you notice that.