First Name:____________________ **Last Name:**____________________

OSU Number:____________________ **Signature:**____________________

OKLAHOMA STATE UNIVERSITY
Department of Mathematics

MATH 4613/5003 (Modern Algebra I)
Instructor: Dr. Mathias Schulze

MIDTERM 2
October 27, 2010

Duration: 50 minutes

No aids allowed.

This examination paper consists of **5** pages and **3** questions. Please bring any discrepancy to the attention of an invigilator. The number in brackets at the start of each question is the number of points the question is worth.

Answer **3** questions.

To obtain credit, you must give arguments to support your answers.

For graders' use:

	Score
1 (15)	
2 (15)	
3 (15)	
Total (45)	

1. [15] Formulate definitions (describe the data involved and list the defining properties), give examples or answer questions without proof.

   (a) For a subset $A \subset G$ of a group G, define $\langle A \rangle$.

   (b) What is a composition series?

   (c) Define the alternating group A_n.

   (d) For $N \trianglelefteq G$, define the group G/N (underlying set and group operation).

   (e) What is a simple group?

   Solution: See textbook and lecture notes.

2. [15] Formulate results (list all hypotheses and formulate the statement, no proofs), answer questions without proofs.

   (a) Sylow's Theorem (about subgroups of groups of order $p^k m$ with $(p, m) = 1$).

   (b) First Isomorphism Theorem.

   (c) Second (Diamond) Isomorphism Theorem.

   (d) Jordan–Hölder Theorem (about composition series).

   (e) Feit–Thompson Theorem (about simple groups of odd order).

   Solution: See textbook and lecture notes.

3. [15] Prove statements (give rigorous arguments based on the definitions). Pick 3 out of 5 subproblems.

   (a) If $H \leq G$ and $K \leq G$ have coprime orders then $H \cap K = 1$.

   (b) Let $M \trianglelefteq G$ and $N \trianglelefteq G$ and $G = MN$; show that $G/(M \cap N) \cong (G/M) \times (G/N)$.

   (c) Prove that S_n is generated by transpositions $(i, i+1)$, $i = 1, \ldots, n-1$.

   (d) Find all finite groups which have exactly two conjugacy classes.

   (e) Prove that every non-Abelian group of order 6 has a non-normal subgroup of order 2, and conclude that every such group is isomorphic to S_3.

Solution:

(a) By Lagrange's Theorem, $|H \cap K| \mid |H|$ and $|H \cap K| \mid |K|$. As $|H|$ and $|K|$ are coprime, it follows that $|H \cap K| = 1$ and hence $H \cap K = 1$.

(b) The kernel of the homomorphism $G \to G/M \times G/N$ is $M \cap N$. To see that it is surjective, let $(\overline{n}, \overline{m}) \in G/M \times G/N$. By assumption $G = MN$, and $G = NM$ as M is normal. So we can choose $n \in N$ and $m \in M$. As M is normal, $m = m'^n$ for some $m' \in M$, and hence $mn = nm'$ is a preimage. Now apply the first isomorphism theorem.

(c) See hint in Exercise 3.5.3.

(d) Pick $h \in G \setminus \{1\}$. Then conjugation $g \mapsto h^g$ defines a surjection, and hence a bijection, $G \setminus \{1\} \to G \setminus \{1\}$. So $C_G(h) = \{1, h\}$ and the class equation reads $|G| = 1 + [G : C_G(h)] = 1 + \frac{|G|}{2}$. Thus, $|G| = 2$ and hence $G \cong Z_2$.

(e) As G is not Abelian, $Z(G) = 1$ by Exercise 3.1.36 and hence $C_G(x) = \langle x \rangle$ for all $1 \neq x \in G$. So the class equation reads $6 = |G| = 1 + 2k + 3l$ where k and l are the numbers of order 3 and 2 elements respectively. It follows that $k = 1 = l$. Pick $x, y \in G$ with $|x| = 3$ and $|y| = 2$. Assuming that $H = \langle y \rangle \trianglelefteq G$ gives $y^x = y$, for all $x \in G$, and then G Abelian, in contradiction to the assumption. So H is not normal and hence $H \cap H^x = 1$ for some $x \in G$. Then $G \hookrightarrow S_{G/H} \cong S_3$ by Theorem 4.2.3.(3), and hence $G \cong S_3$.

End of examination
Total pages: 5
Total marks: 45