HOMEWORK 2

Define your terminology and explain notation. If you require a standard result, such as one of the Sylow theorems, then state it (or cite a result from your textbook) before you use it; otherwise give clear and complete proofs of your claims. The problems are of equal value, 5 points each. Partial solutions will be considered on their merits.

Exercise 1. Let G be a finite non-abelian group and let p be the smallest prime divisor of $|G|$. Then $[G : Z_G] \geq p^2$. (Hint: Consider $C_G(x)$ for a suitable $x \in G$.)

Exercise 2. Let $N \trianglelefteq G$ with $\operatorname{Aut} N = \operatorname{Inn} N$ and $Z_N = 1$, and set $H = C_G(N)$. Show that $G \cong N \times H$. (Hint: Show that for each $g \in G$ there is an $n \in N$ such that $x^g = x^n$ for all $x \in N$. Deduce that $n^{-1}g \in H$.)

Exercise 3. Let G be a group.

(a) If G is solvable then $G^{(k)} = 1$ for $k \gg 0$.

(b) Show that $G' \leq \langle x^2 \mid x \in G \rangle \trianglelefteq G$. Give an example where the first inclusion is strict.

Exercise 4.

(a) If G is finite cyclic then $[G : G^m] = (|G|, m)$.

(b) If G_1 and G_2 are cyclic of coprime orders then $G_1 \times G_2$ is cylcic.

Exercise 5.

(a) Let $1 \to G' \to G \to G'' \to 1$ be an exact sequence of groups. Show that G is solvable if and only if G' and G'' are solvable.

(b) Read the Jordan–Hölder Theorem in your text book (see Ch. I, §4, Thm. 3.5). Show by example that the statement is wrong if the word "simple" is replaced by the word "abelian" or "cyclic".

Exercise 6.

(a) Explicitly construct a subgroup H of $G = Z_{27} \times Z_3$ such that $G/H \cong Z_9$.

(b) Find all subgroups H of $G = Z_8 \times Z_2$ and express the quotients G/H as direct products of cyclic groups of prime power order.

Exercise 7. Let G be a finite group acting on a finite set S. For $x \in G$, denote by $S^x = \{s \in S \mid xs = s\}$ the fixed point set of x.

(a) Show that $\sum_{x \in G} |S^x| = \sum_{s \in S} |G_s|$. (Hint: Compare the definitions of S^x and G_s.)

(b) Deduce that $|\{Gs \mid s \in S\}| = \frac{1}{|G|} \sum_{x \in G} |S^x|$. (This means that the number of orbis equals the average number of fixed points.)

Exercise 8. Let G be a finite group. Show that $H \lneq G$ implies that $\bigcup_{x \in G} H^x \neq G$. (Hint: Apply Exercise 7.(b) to the action $G \to \operatorname{Perm}(G/H)$).