HOMEWORK 3

Define your terminology and explain notation. If you require a standard result, such as one of the Sylow theorems, then state it (or cite a result from your textbook) before you use it; otherwise give clear and complete proofs of your claims. The problems are of equal value, 5 points each. Partial solutions will be considered on their merits.

Exercise 1. Let $\sigma = (1\,2\,3 \cdots n) \in S_n$.

(a) Show that the conjugacy class of σ has $(n-1)!$ elements.

(b) Show that $C_{S_n}(\sigma) = \langle \sigma \rangle$.

Exercise 2. Show that S_4 is solvable and list its composition factors.

Exercise 3. Show that any subgroup of S_7 of order $6!$ is isomorphic to S_6.

Exercise 4. Let G be an abelian group.

(a) Show that, for any group H, the set $\mathrm{Hom}(H, G)$ of all group homomorphisms $H \to G$ has a natural group structure. (Hint: Recall how you add functions in Calculus.)

(b) Show that $\mathrm{Hom}(\mathbb{Z}, G) \cong G$ for any group G.

(c) Identify $\mathrm{Hom}(\mathbb{Z}/k\mathbb{Z}, G)$ with a subgroup of G.

Exercise 5. Prove that any group of order 992 is solvable.

Exercise 6.

(a) Show that $\mathrm{Aut}(Z_p) \cong Z_{p-1}$ for any prime p.

(b) Show that there is up to isomorphism a unique group of order 35, and that it is abelian.

Exercise 7. Let G be a group of order 175. Show that the 7-Sylow group is contained in the center.

Exercise 8. Show that there are exactly two isomorphism classes of groups of order 55 and describe them by generators and relations.