# HOMEWORK 4

Define your terminology and explain notation. If you require a standard result, such as one of the Sylow theorems, then state it (or cite a result from your textbook) before you use it; otherwise give clear and complete proofs of your claims. The problems are of equal value, 6 points each. Partial solutions will be considered on their merits.

**Exercise 1.** Let $G$ be a finite $p$-group and $H$ be a nontrivial normal subgroup of $G$. Show that $1 \neq H \cap Z_G$.

**Exercise 2.** Let $H$ be a normal subgroup of $G$, such that $|H|$ is a prime $p$. Show that $H$ is contained in every $p$-Sylow subgroup of $G$.

**Exercise 3.** Let $G$ be a group with $|G| = rpq$, where $r < p < q$ are primes. Show that:

(a) Any $q$-Sylow subgroup is normal.

(b) $G$ has a normal subgroup $H$ of index $r$.

(c) If $p \nmid q - 1$, then any $p$-Sylow subgroup is normal.

**Exercise 4.** For $p$ and $q$ distinct primes and $|H| = pq$, give a formula for $\mathrm{Aut}(H)$ in terms of cyclic groups.

**Exercise 5.** Use the preceeding two exercises to classify all groups of order 30. (Write your result in terms of cyclic and dihedral groups.)

**Exercise 6.** Determine all groups of order $\leq 10$ up to isomorphism.