HOMEWORK 2

Define your terminology and explain notation. If you require a standard result, such as one of the Sylow theorems, then state it (or cite a result from your textbook) before you use it; otherwise give clear and complete proofs of your claims. The problems are of equal value, 6 points each. Partial solutions will be considered on their merits.

Exercise 1. If α is a complex root of $X^6 + X^3 + 1$, find all homomorphisms $\mathbb{Q}(\alpha) \to \mathbb{C}$. Hint: The polynomial divides $X^9 - 1$.

Exercise 2. Let α be a real root of $X^4 - 5$. Show that:

(a) $\mathbb{Q}(i\alpha^2)/\mathbb{Q}$ is normal.

(b) $\mathbb{Q}(\alpha + i\alpha)/\mathbb{Q}(i\alpha^2)$ is normal.

(c) $\mathbb{Q}(\alpha + i\alpha)/\mathbb{Q}$ is not normal.

Exercise 3.

(a) If the roots of a monic polynomial $f \in F[X]$ are distinct, and form a field, then $\operatorname{char} F = p$ and $f = X^{p^n} - X$ for some $n \geq 1$.

(b) Let K be a field with p^n elements. Show that every element of K has a unique p-th root in K.

Exercise 4. Let $\operatorname{char} K = p$ and $[L\colon K] < \infty$ coprime to p. Show that L/K is separable.

Exercise 5. Let $\operatorname{char} K = p$ and α algebraic over K. Show that α is separable if and only if $K(\alpha) = K(\alpha^{p^n})$ for all $n \geq 1$.

Exercise 6. Show that every element of a finite field can be written as a sum of two squares in that field.