TECHNISCHE UNIVERSITÄT KAISERSLAUTERN

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Computer Algebra

Winter Semester 2014 - Problem Set 9

Due January 15, 2015, 10:00

Problem 1:

(a) Let K be a field of characteristic 0, $\overline{K}$ its algebraic closure and $\mathfrak{a} \trianglelefteq K[x_1, \ldots, x_n]$ an ideal. Prove that $\mathfrak{a} \cdot \overline{K}[x_1, \ldots, x_n] \cap K[x_1, \ldots, x_n] = \mathfrak{a}$

(b) Let R be a Noetherian ring, $\mathfrak{a} \trianglelefteq R$ an ideal. Show that the two conditions are equivalent:

(i) $\operatorname{Ass}_R(R/\mathfrak{a}) = \{\mathfrak{p}\}$ for some $\mathfrak{p} \in \operatorname{Spec}(R)$

(ii) for any $a, b \in R$, $ab \in \mathfrak{a}$ and $a \notin \mathfrak{a}$ implies $b \in \sqrt{\mathfrak{a}}$

If the above conditions are satisfied, we have $\mathfrak{p} = \sqrt{\mathfrak{a}}$.

Problem 2: Let $\mathfrak{a} \trianglelefteq K[x_1, \ldots, x_n]$, $\mathfrak{a} = \mathfrak{q}_1 \cap \ldots \cap \mathfrak{q}_r$ be an irredundant primary decomposition. Let $u \subseteq \{x_1, \ldots, x_n\}$ be an independent set with respect to $\mathfrak{a}$. Suppose that $\mathfrak{q}_i \cap K[u] = \langle 0 \rangle$ for $0 < i \leq s$ and that $\mathfrak{q}_i \cap K[u] \neq \langle 0 \rangle$ for $s < i \leq r$ for some $1 \leq s \leq r$. Prove that $\mathfrak{a} \cdot K(u)[x \setminus u] = \bigcap_{i=1}^{s} \mathfrak{q}_i \cdot K(u)[x \setminus u]$ is an irredundant primary decomposition.

Problem 3: Let $>$ be a monomial ordering on $\operatorname{Mon}(x_1, \ldots, x_n)$, let $\mathfrak{a} \trianglelefteq K[x_1, \ldots, x_n]$ an ideal. Let $u \subseteq \{x_1, \ldots, x_n\}$ be an independent set with respect to $\operatorname{LM}_>(\mathfrak{a})$. Prove that u is an independent set with respect to I. Use the fact that

$$\dim(K[x_1, \ldots, x_n]/\mathfrak{a}) = \dim(K[x_1, \ldots, x_n]/\operatorname{LM}_>(\mathfrak{a}))$$

to see that a maximal independent set for $\operatorname{LM}_>(\mathfrak{a})$ is also a maximal independent set for $\mathfrak{a}$.

Problem 4 Modify the procedure in the SINGULAR Example 3.5.9 in [GP][1] to compute a maximal independent set for an ideal.

[1] Gert-Martin Greuel, Gerhard Pfister: "A SINGULAR Introcution to Commutative Algebra"