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Algebraic Geometry

Summer Semester 2015 - Problem Set 10

Due June 26, 2015, 11:00 am

In all exercises, the ground field k is assumed to be algebraically closed.

Problem 1. Let X be an affine irreducible variety. Show:

(a) $K(X)$ is isomorphic to the quotient field of $A(X)$.

(b) $\mathscr{O}_{X,a}$ is naturally a subring of $K(X)$ for $a \in X$.

Problem 2. Let $X \subset \mathbb{P}^n$ be a quadric, i.e. an irreducible variety which is the zero locus of an irreducible homogeneous polynomial of degree 2. Show that X is birational to $\mathbb{P}^{n-1}$, but X is not ismorphic to $\mathbb{P}^{n-1}$ in general.

Problem 3. Let $\operatorname{char}(k) \neq 2$. Recall that a projective conic in $\mathbb{P}^2$ is the zero locus of an irreducible homogeneous polynomial of degree 2 in $k[x_0, x_1, x_2]$.

(a) Considering the coefficients of such polynomials, show that the set of all conics in $\mathbb{P}^2$ can be identified with an open subset of the projective space $\mathbb{P}^5$.

(b) Let $a \in \mathbb{P}^2$. Show that the subset of U consisting of all conics in $\mathbb{P}^2$ passing through a is the zero locus of a linear equation in the homogeneous coordinates of $U \subset \mathbb{P}^5$.

(c) Given 5 points in $\mathbb{P}^2$, no three of which lie on a line, show that there is a unique conic in $\mathbb{P}^5$ passing through all these points.

Problem 4. Let $X \subset \mathbb{P}^3$ be the degree-3 Veronese embedding of $\mathbb{P}^1$, i.e. the image of the morhpism $\mathbb{P}^1 \to \mathbb{P}^3$, where $(x_0 : x_1) \mapsto (x_0^3 : x_0^2 x_1 : x_0 x_1^2 : x_1^3)$.

Moreover, let $a = (0 : 0 : 1 : 0) \in \mathbb{P}^3$ and $L = V(y_2) \subset \mathbb{P}^3$, and consider the projection f from a to L as in Example 7.6 (b).

(a) Show that f is a morphism.

(b) Determine an equation for the curve $f(X)$ in $L \cong \mathbb{P}^2$.

(c) Is $f : X \to f(X)$ an isomorphism onto its image?