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Algebraic Geometry

Summer Semester 2015 - Problem Set 4

Due Mai 15, 2015, 11:00 am

In all exercises, the ground field k is assumed to be algebraically closed.

Problem 1.

(a) Let $X \subset \mathbb{A}^3$ be the union of the three coordinate axes. Determine generators for the ideal $I(X)$. Show that $I(X)$ cannot be generated by fewer than 3 elements and that X has dimension 1.

(b) Let $X = \{(t, t^3, t^5) \mid t \in k\} \subset \mathbb{A}^3$. Show that X is an affine variety of dimension 1 and compute $I(X)$.

Problem 2. In the lecture, we have seen (without proof), that $\dim(\mathbb{A}^n) = n$. The aim of this problem is to establish this result in case $n = 2$. Let $X \subset \mathbb{A}^2$ be an irreducible algebraic variety. Show that either

- $X = Z(0)$, i.e. X is the whole space $\mathbb{A}^2$, or
- $X = Z(f)$ for some irreducible polynomial f in $k[x, y]$, or
- $X = Z(x - a, y - b)$ for some $a, b \in k$, i.e. X is a single point.

Deduce that $\dim(\mathbb{A}^2) = 2$.

Hint: Show that the common zero locus of two polynomials $f, g \in k[x, y]$ without common factor is finite using the Gauss Lemma.

Problem 3.

(a) Let $\emptyset \neq X$ be an irreducible affine variety, $f_1, ..., f_r \in A(X)$ and Y an irreducible component of $V(f_1, ..., f_r)$. Prove that $\dim(Y) \geq \dim(X) - r$. Now assume additionally that X is irreducible. Formulate conditions for $f_1, ..., f_r$ such that equality holds.

(b) Let $\emptyset \neq X, Y$ irreducible affine varieties. Prove that $\dim(X \times Y) = \dim(X) + \dim(Y)$.

Problem 4. Are the following statements true or false: if $f : \mathbb{A}^n \to \mathbb{A}^m$ is a polynomial map (i.e. $f(P) = (f_1(P), \ldots, f_m(P))$ with $f_i \in k[x_1, \ldots, x_n]$), and ...

(a) $X \subset \mathbb{A}^n$ is an affine algebraic variety, then the image $f(X) \subset \mathbb{A}^m$ is an affine algebraic variety.

(b) $X \subset \mathbb{A}^m$ is an affine algebraic variety, then the inverse image $f^{-1}(X) \subset \mathbb{A}^n$ is an affine algebraic variety.

(c) $X \subset \mathbb{A}^n$ is an affine algebraic variety, then the graph $\Gamma = \{(x, f(x)) \mid x \in X\} \subset \mathbb{A}^{n+m}$ is an affine algebraic variety.