TECHNISCHE UNIVERSITÄT KAISERSLAUTERN

Prof. Dr. Mathias Schulze
B.Sc. Lucas Ruhstorfer
Fachbereich Mathematik

Algebraic Geometry

Summer Semester 2015 - Problem Set 6

Due May 29, 2015, 11:00 am

In all exercises, the ground field k is assumed to be algebraically closed.

Problem 1. Let $f : X \to Y$ be a morphism of affine varieties and $f^* : A(Y) \to A(X)$ the corresponding homomorphism of the coordinate rings. Are the following statements true or false?

(a) f is surjective if and only if f^* is injective.

(b) f is injective if and only if f^* is surjective.

(c) Let $f : \mathbb{A}^1 \to \mathbb{A}^1$ be an isomorphism, then f is of the form $f(x) = ax+b$ for some $a, b \in \mathbb{A}^1$.

(d) Let $f : \mathbb{A}^2 \to \mathbb{A}^2$ be an isomorphism, then f is of the form $f(x) = ax + b$ for some 2×2 matrix $A \in \mathrm{Mat}(2 \times 2, k)$ and $b \in k^2$.

Problem 2. Which of the following ringed spaces are isomorphic over $k = \mathbb{C}$?

(a) $\mathbb{A}^1 \setminus \{1\}$

(b) $V(x_1^2 + x_2^2) \subset \mathbb{A}^2$

(c) $V(x_2 - x_1^2, x_3 - x_1^3) \setminus \{0\} \subset \mathbb{A}^3$

(d) $V(x_1 x_2) \subset \mathbb{A}^2$

(e) $V(x_2^2 - x_1^3 - x_1^2) \subset \mathbb{A}^2$

(f) $V(x_1^2 - x_2^2 - 1) \subset \mathbb{A}^2$

Problem 3. Show:

(a) Every morphism $f : \mathbb{A}^1 \setminus \{0\} \to \mathbb{P}^1$ can be extended to a morphism $f : \mathbb{A}^1 \to \mathbb{P}^1$.

(b) Not every morphism $f : \mathbb{A}^2 \setminus \{0\} \to \mathbb{P}^1$ can be extended to a morphism $f : \mathbb{A}^2 \to \mathbb{P}^1$.

(c) Every morphism $f : \mathbb{P}^1 \to \mathbb{A}^1$ is constant.

Problem 4. Prove:

(a) Every isomorphism $f : \mathbb{P}^1 \to \mathbb{P}^1$ is of the form $f(x) = \frac{ax+b}{cx+d}$ for some a, b, c, d, where x is an affine coordinate on $\mathbb{A}^1 \subset \mathbb{P}^1$.

(b) Given three distinct points $a_1, a_2, a_3 \in \mathbb{P}^1$ and three distinct points $b_1, b_2, b_3 \in \mathbb{P}^1$, show that there is a unique isomorphism $f : \mathbb{P}^1 \to \mathbb{P}^1$ such that $f(a_i) = b_i$ for $i = 1, 2, 3$.