

Computer Algebra

Winter Semester 2015 - Problem Set 6

Due December 10, 2015, 10:00

Problem 1: (*Computing ideal quotients*) Let K be a field of characteristic 0 and let $>$ be a monomial ordering on $\text{Mon}(x_1, \dots, x_n)$. Consider the two ideals $\mathfrak{a}, \mathfrak{b} \trianglelefteq K[x_1, \dots, x_n]_{>}$ and assume that $\mathfrak{a} = \langle g_1, \dots, g_r \rangle$, $\mathfrak{b} = \langle h_1, \dots, h_s \rangle$, where $g_i, h_j \in K[x_1, \dots, x_n]$. Define $h := h_1 + t \cdot h_2 + \dots + t^{s-1} h_s \in K[t, x_1, \dots, x_n]$. Prove that

$$\mathfrak{a} : \mathfrak{b} = \langle \langle g_1, \dots, g_r \rangle_{K[t, x_1, \dots, x_n]} : h \rangle \cap K[x_1, \dots, x_n]_{K[x_1, \dots, x_n]_{>}}.$$

Problem 2: (*Zariski closure of the image*) Consider $\varphi : \mathbb{Q}^2 \rightarrow \mathbb{Q}^4$, $(s, t) \mapsto (s^4, s^3t, st^3, t^4)$. Compute the Zariski closure of the image, $\overline{\varphi(\mathbb{Q}^2)}$, and decide whether $\varphi(\mathbb{Q}^2)$ coincides with its closure.

Problem 3: Let $>$ be a monomial ordering on $\text{Mon}(x_1, \dots, x_n)$, and let $I \subseteq K[x_1, \dots, x_n]$ be an ideal with primary decomposition $I = \bigcap_{i=1}^t Q_i$. Show that

$$I \cdot R \cap K[x_1, \dots, x_n] = \bigcap_{i: Q_i R \neq R} Q_i,$$

where $R := K[x_1, \dots, x_n]_{>}$.

Problem 4: Change your SINGULAR procedure computing a Gröbner basis in such a way that

1. the pair set P is sorted in ascending order w.r.t. $\text{lcm}(\text{LM}_{>}(f_1), \text{LM}_{>}(f_2))$, $f_1, f_2 \in P$.
2. it takes an optional parameter such that if this optional parameter is the string “minimal”, the procedure returns a minimal Gröbner basis, and if this optional parameter is missing, the procedure just returns some standard basis as before.

HINT: If you add `list #` at the end of the input of your procedure, then your procedure allows for any number of optional parameters. You can test with `size(#)` the number of optional parameters the user has provided, while `#[i]` gives you the i -th optional parameter.