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Commutative Algebra

Winter Semester 2016 - Problem Set 12

Due February 3, 2017, 1 p.m.

Problem 1: Let R be a reduced ring with $|\operatorname{Min}\operatorname{Spec}(R)| < \infty$. Show that R is normal in the sense of Definition 7.23 if and only if R is normal in the sense of Remark 7.29.

Problem 2: Let K be an infinite field, $\underline{X} = (X_1, \ldots, X_n)$. For $a = (a_1, \ldots, a_{n-1}) \in K^{n-1}$ consider the automorphism of K-algebras

$$\varphi_a \colon K[\underline{X}] \to K[\underline{X}], \quad X_n \mapsto X_n, \quad X_i \mapsto X_i + a_i X_n,\ i = 1, \ldots, n-1.$$

Show that for $f \in K[\underline{X}]$ there exists some $a \in K^{n-1}$ such that $\varphi_a(f) = cX_n^m +$ terms of smaller X_n-degree for some $0 \neq c \in K$ and $m \in \mathbb{N}$.

Problem 3: Let R and S be affine K-algebras. Show that

$$\dim(R \otimes_K S) = \dim(R) + \dim(S).$$

Hint: Consider the cocartesian diagram obtained by tensoring Noether normalizations of R and S.

Problem 4: Let R be a ring, $J \trianglelefteq R$, $I/J \trianglelefteq R/J$, $V \subseteq Z_R(J)$ and write $V/J := \{\mathfrak{m}/J \mid \mathfrak{m} \in V\}$. Show that

$$Z_{R/J}(I/J) = Z_R(I)/J, \quad I_{R/J}(V/J) = I_R(V)/J.$$