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Commutative Algebra

Winter Semester 2016 - Problem Set 3

Due November 18, 2016, 1 p.m.

Problem 1:

(a) Show that commutative diagrams of R-modules with exact upper row and lower row a complex as below extend by a unique dashed arrow.

$$
\begin{array}{ccccccc}
0 & \longrightarrow & K & \xrightarrow{\iota} & N & \xrightarrow{\varphi} & M \\
 & & \uparrow{\scriptstyle \exists^1 \kappa_{(\alpha,\beta)}} & & \uparrow{\scriptstyle \alpha} & & \uparrow{\scriptstyle \beta} \\
 & & K' & \xrightarrow{\iota'} & N' & \xrightarrow{\varphi'} & M'
\end{array}
\qquad
\begin{array}{ccccccc}
N & \xrightarrow{\varphi} & M & \xrightarrow{\pi} & C & \longrightarrow & 0 \\
\downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \exists^1 \gamma_{(\alpha,\beta)}} & & \\
N' & \xrightarrow{\varphi'} & M' & \xrightarrow{\pi'} & C' & &
\end{array}
$$

(b) Show that ker is a functor mapping an R-linear map $\varphi \in \mathrm{Hom}_R(N, M)$ to $\ker(\varphi)$ and a morphism of R-linear maps $(\alpha, \beta) : (\varphi' : N' \to M') \to (\varphi : N \to M)$ to $\kappa_{(\alpha,\beta)} : \ker(\varphi') \to \ker(\varphi)$, where $\kappa_{(\alpha,\beta)}$ is defined by the left commutative diagram in part (a). I.e. show that:

(i) $\kappa_{(\mathrm{id}_N,\mathrm{id}_M)} = \mathrm{id}_{\ker(\varphi)}$ for $(\mathrm{id}_N, \mathrm{id}_M) : (\varphi : N \to M) \to (\varphi : N \to M)$

(ii) $\kappa_{(\alpha,\beta)\circ(\alpha',\beta')} = \kappa_{(\alpha,\beta)} \circ \kappa_{(\alpha',\beta')}$ for $(\alpha', \beta') : (\varphi'' : N'' \to M'') \to (\varphi' : N' \to M')$ and $(\alpha, \beta) : (\varphi' : N' \to M') \to (\varphi : N \to M)$

(c) Show that cok, im and H_i are functors.

(d) (In-class) Are the above functors additive?

Problem 2: *You only need to hand in part (a) and the first part of (c) (i.e. a proof of split exactness of the middle sequence in (c)) of the following problem:*

(a) A sequence $0 \longrightarrow M' \xrightarrow{\alpha} M \xrightarrow{\beta} M''$ of R-modules is exact if and only if for all R-modules N the induced sequence

$$0 \longrightarrow \mathrm{Hom}_R(N, M') \xrightarrow{\alpha_*} \mathrm{Hom}_R(N, M) \xrightarrow{\beta_*} \mathrm{Hom}_R(N, M'')$$

is exact.

(b) A sequence $M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \longrightarrow 0$ of R-modules is exact if and only if for all R-modules N the induced sequence

$$\mathrm{Hom}_R(M', N) \xleftarrow{\alpha^*} \mathrm{Hom}_R(M, N) \xleftarrow{\beta^*} \mathrm{Hom}_R(M'', N) \longleftarrow 0$$

is exact.

(c) For any split exact sequence $0 \longrightarrow M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \longrightarrow 0$ of R-modules also

$$0 \longrightarrow \mathrm{Hom}_R(N, M') \xrightarrow{\alpha_*} \mathrm{Hom}_R(N, M) \xrightarrow{\beta_*} \mathrm{Hom}_R(N, M'') \longrightarrow 0\,,$$

$$0 \longleftarrow \mathrm{Hom}_R(M', N) \xleftarrow{\alpha^*} \mathrm{Hom}_R(M, N) \xleftarrow{\beta^*} \mathrm{Hom}_R(M'', N) \longleftarrow 0$$

are split exact. □

Problem 3: Let P be an R-module. Show that the following are equivalent:

(a) $\operatorname{Hom}_R(P, -)$ is exact, that is, preserves exact sequences.

(b) Any exact sequence $0 \to N \to M \to P \to 0$ splits.

(c) $M \oplus P$ is free for some R-module M.

Problem 4: Let M be an R-module and $\mathcal{G}_R(M) := \{E \subseteq M \mid \langle E \rangle = M\}$ the set of generating systems of M. A *minimal system of generators* for M is a minimal element of $\mathcal{G}_R(M)$ with respect to inclusion. We define $\mu_R(M) := \min\{|E| \mid E \in \mathcal{G}_R(M)\}$. Give examples of the following:

(a) An R-module M and a generating system $E \in \mathcal{G}_R(M)$ with $|E| = \mu_R(M)$ which is not minimal.

(b) A proper R-submodule $M' \subsetneq M$ with $\mu_R(M') > \mu_R(M)$.

(c) Two R-modules M, M' with $\mu_R(M \oplus M') < \mu_R(M) + \mu_R(M')$.

(d) (Extra Credit) An R-module which does not have a minimal generating system. (*Hint: Consider the $\mathbb{Z}$-module $\mathbb{Q}$ and use that $\mathbb{Q}$ is a divisible group.*)