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Commutative Algebra

Winter Semester 2016 - Problem Set 5

Due December 2, 2016, 1 p.m.

Problem 1:

(a) Let K be a field, $R = K[x, y, z]/\langle xz, yz\rangle$ and $\mathfrak{p} = \langle \overline{x}, \overline{y}, \overline{z-1}\rangle \trianglelefteq R$. Show that $\mathfrak{p}$ is a prime ideal and that $R_\mathfrak{p} \cong K[z]_{\langle z-1\rangle}$.

(b) Let R be a ring and $f \in R$. Show that $R_f \cong R[x]/\langle fx - 1\rangle$.

Problem 2: Let R be a ring. Show:

(a) Let $\varphi \in \operatorname{Hom}(R, S)$, $U \subseteq R$ multiplicatively closed, N an S-module. Then $\varphi(U) \subseteq S$ is multiplicatively closed and $U^{-1}N \cong \varphi(U)^{-1}N$ as modules over $U^{-1}S \cong \varphi(U)^{-1}S$.

(b) Let M and N be R-modules and $U \subseteq R$ multiplicatively closed. If M is finitely presented then

$$U^{-1} \operatorname{Hom}_R(M, N) \cong \operatorname{Hom}_{U^{-1}R}(U^{-1}M, U^{-1}N).$$

Problem 3: Let R be a ring and U a multiplicatively closed subset of R. Show:

(a) For $I \trianglelefteq R$, $U^{-1}\sqrt{I} = \sqrt{U^{-1}I}$.

(b) $U^{-1}N(R) \cong N(U^{-1}R)$ and $U^{-1}(R^{\text{red}}) \cong (U^{-1}R)^{\text{red}}$.

(c) “Being reduced” is a local property, i.e. the following are equivalent:

   (i) R is reduced.

   (ii) $R_\mathfrak{p}$ is reduced for all $\mathfrak{p} \in \operatorname{Spec}(R)$.

   (iii) $R_\mathfrak{m}$ is reduced for all $\mathfrak{m} \in \operatorname{Max}\operatorname{Spec}(R)$.

(d) “Being a domain” is not a local property.

Problem 4: Let U be a multiplicatively closed subset of a ring R. U is called *saturated* if

$$u \cdot u' \in U \Leftrightarrow u \in U \text{ and } u' \in U.$$

The set $\overline{U} := \{r \in R \mid r \cdot r' \in U \text{ for some } r' \in R\}$ is the *saturation* of U.

(a) Show: $U^{-1}R \cong \overline{U}^{-1}R$.

(b) Let V be a multiplicatively closed subset of R such that $U \subseteq V$. Show that the morphism $\varphi : U^{-1}R \to V^{-1}R : r/u \mapsto r/u$ is an isomorphism if and only if $V \subseteq \overline{U}$.

(c) Let $U' \subseteq R$. Prove that U' is multiplicatively closed and saturated if and only if $R \setminus U'$ is a (possibly empty) union of prime ideals.
Hint: For the “only if”-part use the localization map $R \to U'^{-1}R$ to show that for each $r \in R \setminus U'$ there exists a prime ideal $\mathfrak{p} \in D_R(U')$ such that $r \in \mathfrak{p}$.

(d) Let $I \trianglelefteq R$ be an ideal. Determine $\overline{1 + I}$, where $1 + I := \{1 + i \mid i \in I\}$.