TECHNISCHE UNIVERSITÄT KAISERSLAUTERN

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Computer Algebra

Summer Term 2019 - Sheet 10

Due Date: Thursday, June 27, 2019, 10:00 am

Exercise 33. Let $>$ be a local ordering, $A = K[x]_>$ and $I \subseteq A^r$ be a finitely generated A-module. Furthermore, let the matrices $A_1, \ldots, A_m$ be the output of Algorithm 2.5.16 in the book by Greuel and Pfister applied to I.
State an algorithm that computes the Betti numbers of A^r/I without computing a minimal free resolution and prove its correctness.

Our goal in the next two exercises is to compute $\operatorname{Hom}(M, N)$ for finitely presented R-modules M and N. Let R be a ring, M, N be finitely presented R-modules with presentations

$$R^{m_1} \xrightarrow{\psi_1} R^{n_1} \longrightarrow M \longrightarrow 0 \quad \text{and} \quad R^{m_2} \xrightarrow{\psi_2} R^{n_2} \longrightarrow N \longrightarrow 0.$$

Furthermore, let $\varphi \in \operatorname{Hom}(M, N)$ be a homomorphism. Recall the following:

(a) There exist $\varphi_1 \in \operatorname{Hom}(R^{m_1}, R^{m_2})$ and $\varphi_2 \in \operatorname{Hom}(R^{n_1}, R^{n_2})$, such that the following diagram is commutative and has exact rows.

$$\begin{array}{ccccccc} R^{m_1} & \xrightarrow{\psi_1} & R^{n_1} & \longrightarrow & M & \longrightarrow & 0 \\ \downarrow{\scriptstyle \varphi_1} & & \downarrow{\scriptstyle \varphi_2} & & \downarrow{\scriptstyle \varphi} & & \\ R^{m_2} & \xrightarrow{\psi_2} & R^{n_2} & \longrightarrow & N & \longrightarrow & 0. \end{array}$$

(b) $\operatorname{Ker}(\varphi) \cong \varphi_2^{-1}(\operatorname{Im}(\psi_2)) / \operatorname{Im}(\psi_1)$.

Exercise 34. We keep the setup and notation from above. Let A be a $n_1 \times k$ matrix, which has a set of generators of $\operatorname{Ker}\left(R^{n_1} \xrightarrow{\overline{\varphi_2}} R^{n_2}/\operatorname{Im}(\psi_2)\right)$ as columns, where $\overline{\varphi}_2$ is the map $R^{n_1} \xrightarrow{\varphi_2} R^{n_2} \twoheadrightarrow R^{n_2}/\operatorname{Im}(\psi_2)$.
Denote by $\overline{A}$ the map $R^k \xrightarrow{A} R^{n_1} \twoheadrightarrow R^{n_1}/\operatorname{Im}(\psi_1)$.
Prove that

$$\operatorname{Ker}(\varphi) \cong R^k / \operatorname{Ker}\left(R^k \xrightarrow{\overline{A}} R^{n_1}/\operatorname{Im}(\psi_1)\right).$$

Exercise 35. Let R be a ring, M, N be finitely presented R-modules with presentations

$$R^{m_1} \xrightarrow{\psi_1} R^{n_1} \longrightarrow M \longrightarrow 0 \quad \text{and} \quad R^{m_2} \xrightarrow{\psi_2} R^{n_2} \longrightarrow N \longrightarrow 0.$$

State an algorithm to compute $\operatorname{Hom}(M, N)$ and prove its correctness.
Hint: See Example 2.1.26 in the book by Greuel and Pfister.