TECHNISCHE UNIVERSITÄT KAISERSLAUTERN

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Plane Algebraic Curves

Summer Term 2019 - Problem Set 2

Due Date: Friday, May 10, 2019, 10:00 am

Exercise 1. Let $P \in \mathbb{A}^2$ be a point, and let $F, G \in K[x,y]$ be two non-zero polynomials satisfying $F(P) = G(P) = 0$. Show that:

(a) if F and G are coprime then the family $(F^n)_{n\in\mathbb{N}}$ is linearly independent in $\mathscr{O}_P/\langle G\rangle$,

(b) if F and G have a common factor that vanishes at P, then $\mu_P(F,G) = \infty$.

Exercise 2.

(a) Let us consider the curves $F = y - x^3$ and $G = y^3 - x^4$. Find a polynomial representative of $\frac{1}{x+1}$ in $\mathscr{O}_0/\langle F, G\rangle$, that is to say find a polynomial $f \in K[x,y]$ whose class equals the class of $\frac{1}{x+1}$ in $\mathscr{O}_0/\langle F,G\rangle$.

(b) Prove for arbitrary coprime F and G and any $P \in \mathbb{A}^2$ that every element of $\mathscr{O}_P/\langle F,G\rangle$ has a polynomial representative.

(c) Is this statement true if F and G are not coprime ?

Exercise 3. Let us consider the real curves $F = (x^2+y^2-1)^3 + 10x^2y^2 \in \mathbb{R}[x,y]$ and $G = x^3y^2 - x^2y^2 \in \mathbb{R}[x,y]$.

(a) Let $P = (1,0)$. Compute the intersection multiplicity $\mu_P(F,G)$.

(b) Find all singular points of the curve F and determine the multiplicities and tangents to F at these points.

(c) Show that an irreducible curve F over a field of characteristic 0 has only finitely many singular points

(d) Can you find weaker assumptions on F that also imply that F has only finitely many singular points ?

Exercise 4 (Cusps)**.** Let P be a point on an affine curve F. We say that P is a cusp if $m_P(F) = 2$ and there is exactly one tangent L to F at P, and $\mu_P(F,L) = 3$.

(a) Give an example of a real curve with a cusp, and draw a picture of it.

(b) If F has a cusp at P, prove that F has only one irreducible component passing through P.

(c) If F and G have a cusp at P, what is the minimum possible value for the intersection multiplicity $\mu_P(F,G)$? (*Hint: it can be useful to use the definition of* $\mu_P(F,G)$.)