TECHNISCHE UNIVERSITÄT KAISERSLAUTERN

Prof. Dr. Mathias Schulze
Dr. Delphine Pol
Fachbereich Mathematik

Plane Algebraic Curves

Summer Term 2019 - Problem Set 3

Due Date: Friday, May 24, 2019, 10:00 am

Exercise 1. Let $f : \mathbb{P}^n \to \mathbb{P}^n$ be a *projective coordinate transformation*, that is to say a map of the form:

$$f : (x_0 : \cdots : x_n) \mapsto \big(f_0(x_0, \ldots, x_n) : \cdots : f_n(x_0, \ldots, x_n)\big)$$

for linearly independent homogeneous linear polynomials $f_0, \ldots, f_n \in K[x_0, \ldots, x_n]$.

(a) Let $P_1, \ldots, P_{n+2} \in \mathbb{P}^n$ be points such that any $n+1$ of them are linearly independent in K^{n+1}, and similarly let $Q_1, \ldots, Q_{n+2} \in \mathbb{P}^n$ be points such that any $n+1$ of them are linearly independent. Show that there is a projective coordinate transformation f with $f(P_i) = Q_i$ for all $i \in \{1, \ldots, n+2\}$.

(b) Let F and G be two smooth real projective conics with non-empty set of points. Show that there is a projective coordinate transformation of $\mathbb{P}^2$ that takes F to G.

Exercise 2.

(a) If F, G are polynomials such that $F|G$ and G is homogeneous, then F is homogeneous.

(b) Prove that for a plane curve $F \in K[x, y]$ the number of tangents of F at $P \in F$ is at most $m_P(F)$. Prove that if K is algebraically closed, then the equality holds.

Exercise 3 (Hessian)**.** Let F be a projective curve in the homogeneous coordinates x_0, x_1, x_2. The Hessian associated with F is $H_F := \det \left(\frac{\partial^2 F}{\partial_{x_i} \partial_{x_j}}\right)_{i,j=0,1,2}$ considered as an element of $K[x_0, x_1, x_2]/K^*$.

(a) Show that the Hessian is compatible with projective coordinate transformations (as in Exercise 1), that is to say, if a projective coordinate transformation takes F to F' then it takes H_F to $H_{F'}$.

(b) Let $P \in F$ be a smooth point, and assume that the characteristic of K is 0. Show that $H_F(P) = 0$ if and only if $\mu_P(F, T_pF) \geqslant 3$. Such a point is called an *inflection point* of F.

Hint: By (1)*, you may assume after a coordinate transformation that* $P = (0 : 0 : 1)$ *and* $T_pF = x_1$.

Exercise 4. The following formula is known as Bézout's Theorem:
If F and G are two projective curves without common component over an infinite field then

$$\sum_{P \in F \cap G} \mu_P(F, G) \leqslant \deg(F) \cdot \deg(G), \tag{1}$$

with equality if the ground field is algebraically closed.

For the following complex affine curves F and G, determine the points at infinity of their projective closures, and use (1) to read off the intersection multiplicities at all points of $F \cap G$.

(a) $F = x + y^2$ and $G = x + y^2 - x^3$.

(b) $F = y^2 - x^2 + 1$ and $G = (y + x + 1)(y - x + 1)$.