TECHNISCHE UNIVERSITÄT KAISERSLAUTERN

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Plane Algebraic Curves

Summer Term 2019 - Problem Set 5

Due Date: Friday, June 21, 2019, 10:00 am

Exercise 1. Let K be an infinite field (for example algebraically closed). Let $P_1, \dots, P_6 \in \mathbb{P}^2$ be distinct points so that the six lines $\overline{P_1P_2}$, $\overline{P_2P_3}, \dots, \overline{P_5P_6}$, $\overline{P_6P_1}$ (which can be thought of as the sides of the hexagon with vertices $P_1, \dots, P_6$) are also distinct. Let $P = \overline{P_1P_2} \cap \overline{P_4P_5}$, $Q = \overline{P_2P_3} \cap \overline{P_5P_6}$ and $R = \overline{P_3P_4} \cap \overline{P_6P_1}$ be the intersection points of opposite sides of the hexagon.

(a) Let F be an irreducible projective conic passing through $P_1, \dots, P_5$. We assume that $\overline{P_1R}$ is not tangent to F. Let $P_6' = F \cap \overline{P_1R}$ be the other intersection point of F and $\overline{P_1R}$. What can we say about the points $P' = \overline{P_1P_2} \cap \overline{P_4P_5}$, $Q' = \overline{P_2P_3} \cap \overline{P_5P_6'}$ and $R' = \overline{P_3P_4} \cap \overline{P_6'P_1}$? Show that $\overline{PR} = \overline{P'R'}$.

(b) We assume that P, Q, R lie on a line. Prove that $Q = Q'$ and show that $P_6 = P_6'$. *It gives us the following converse of Pascal's theorem: with the same notations, if P, Q, R lie on a line, then $P_1, \dots, P_6$ lie on a conic.*

Exercise 2 (Cayley-Bacharach Theorem)**.** Let K be an algebraically closed field. Let F and G be two smooth projective cubics. We assume that F and G intersect in exactly 9 distinct points $P_1, \dots, P_9$. Let E be another cubic which contains the points $P_1, \dots, P_8$. We assume that E does not contain P_9. We denote by P_9' the intersection point of E with F which is not in $\{P_1, \dots, P_8\}$.

(a) We assume that L is a line passing through P_9 which does not contain $P_1, \dots, P_8, P_9'$ and which is not tangent to F at any point. We set $H = EL$. Use Noether's Theorem to prove that there exist homogeneous polynomials A and B of degree 1 such that $H = AF + BG$.

(b) By considering the intersection points of L and F, prove that $L = B$.

(c) Deduce the following theorem: if F and G are two smooth projective cubics which intersect in exactly 9 points $P_1, \dots, P_8$ and if E is another cubic containing $P_1, \dots, P_8$, then $P_9 \in E$. *Hint: Show that $P_9' \in B$.*

Exercise 3. Let K be an algebraically closed field and let F be a smooth projective cubic. We assume that L is a line passing transversally through two inflection points P_1 and P_2 of F. We recall from question 3b) of Problem set 3 that P is an inflection point of F if and only if $\mu_P(F, T_PF) \geqslant 3$.

(a) Compute the intersection multiplicity $\mu_P(F, L)$ at each intersection point P of F and L.

(b) Let $H = \prod_{P \in F \cap L} T_pF$. Consider the non reduced curve $G = L^2$. Prove using Noether's Theorem that there exist homogeneous polynomials A and B respectively of degree 0 and 1 such that $H = AF + BG$.

(c) Prove that B contains P_1 and P_2.

(d) Prove that P_3 is also an inflection point of F.

Exercise 4. Let K be an algebraically closed field. Consider the rational function $\varphi = \frac{x^2}{y^2+yz}$ on the projective curve $F = y^2z + x^3 - xz^2$. Let $P = (0 : 0 : 1) \in F$.

(a) Compute the order $n = \mu_P(\varphi)$.

(b) Determine a local coordinate $t \in \mathscr{O}_{F,P}$.

(c) Give an explicit description of φ in the form $\varphi = ct^n$ for some $c \in \mathscr{O}_{F,P}^*$, where c should be written as $\frac{f}{g}$ for some homogeneous $f, g \in S(F)$ with $f(P) \neq 0$ and $g(P) \neq 0$.