Exercise 12. By hand, try to find integers $r$ and $t$ which satisfy the following congruences. Verify your results using SINGULAR.

1. $r \equiv 9t \mod 22$ with $|r| < 9, 0 \leq t \leq \frac{22}{9}$
2. $\gcd(t, 22) = 1$ and $rt^{-1} \equiv 9 \mod 22$ with $|r| < 10, 0 \leq t \leq \frac{11}{9}$
3. $r \equiv 12t \mod 29$ with $|r| < 10, 0 \leq t \leq \frac{29}{10}$
4. $\gcd(t, 29) = 1$ and $rt^{-1} \equiv 12 \mod 29$ with $|r| < 10, 0 \leq t \leq \frac{29}{10}$
5. $r \equiv 13t \mod 36$ with $|r| < 11, 0 \leq t \leq \frac{36}{11}$
6. $\gcd(t, 36) = 1$ and $rt^{-1} \equiv 13 \mod 36$ with $|r| < 11, 0 \leq t \leq \frac{36}{11}$

Exercise 13. Implement modular versions of the following SINGULAR commands:

- `slimgb()`, `gcd()`, `det()`, `trace()`, `transpose()`, `rank()`

Try to find examples where this is faster. In which cases do modular techniques make sense?

Exercise 14.

1. Let $p$ be a prime number and let $f \in \mathbb{Q}[t]$ be an irreducible polynomial such that the reduction $f_p \in \mathbb{F}_p$ of $f$ modulo $p$ is defined and $\deg(f) = \deg(f_p)$. Let us assume that $f$ is reducible and square-free over $\mathbb{F}_p$, that is, if

$$f_p = \prod_{i=1}^{s} f_{i,p} \in \mathbb{F}_p$$

is the irreducible factorization of $f_p$, then $s > 1$ and the $f_{i,p}$ are pairwise distinct.

Write a SINGULAR procedure which checks the above assumptions for a given prime number $p$ and a given polynomial $f \in \mathbb{Q}[t]$ and computes the partition $(\deg(f_{1,p}), \ldots, \deg(f_{s,p}))$ of $\deg(f)$ (in decreasing order).

2. Write a SINGULAR procedure which counts, for a given polynomial $f \in \mathbb{Q}[t]$ and a given integer $N > 1$, how often each partition appears among the results when the procedure from part (1) is applied to the first $N$ primes satisfying the assumptions from part (1).

Where does the distribution on all possible partitions of $\deg(f)$ tend to as $N$ goes to infinity?
Exercise 15.

(1) Find an example which shows that Arnold’s theorem may fail in the non-graded case.

(2) Find an example where the SINGULAR command \texttt{modStd()} gives a wrong result when all tests (including the pTest) are switched off.

(3) Find an example where the SINGULAR command \texttt{modStd()} gives a wrong result when all tests are switched on.