The X(II)-files

Episode: A rational curve on certain Kummer surfaces

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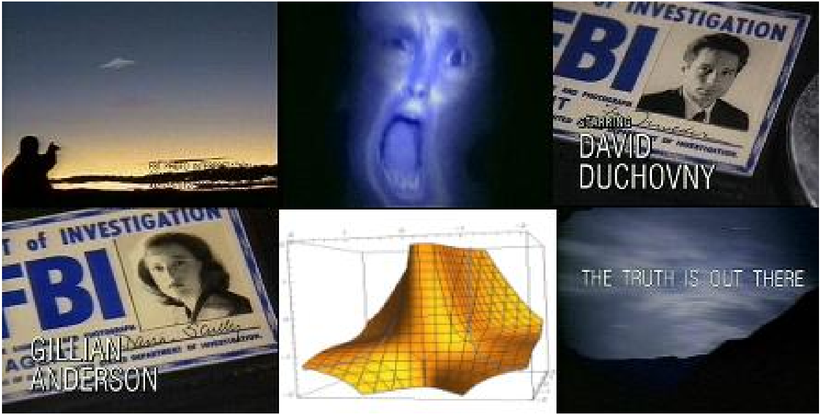
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Introducing the culprit

Take two elliptic curves over $\mathbb{Q}$, not simultaneously of j-invariant 0 or 1728:

$$E_1 : y_1^2 = x^3 + ax + b, \quad a, b \in \mathbb{Q}$$

$$E_2 : y_2^2 = t^3 + ct + e, \quad c, e \in \mathbb{Q}$$

Take the Kummer surface $X := (E_1 \times E_2)/[-1]$. An affine equation for X is

$$(t^3 + ct + e)y^2 = x^3 + ax + b.$$

In a 1993 article, Kuwata and Wang consider a curve on X given by the parametrisation

$$u \mapsto \left(\frac{eu^6 - b}{a - cu^4}, u^3, \frac{eu^6 - b}{u^2(a - cu^4)} \right).$$

This is a source of many rational points.

Who ordered this? There are almost no restrictions on E_1, E_2. (The only other sighting of the curve I could find is a 1992 article by Mestre.)

Taxonomy (i.e. genus)

Denote by C the rational curve on X and by C' its pullback to $A := E_1 \times E_2$. Computations yield:

	$j, j' \notin \{0, 1728\}$		$j = 0$		$j = 1728$
	$j \neq j'$	$j = j'$	$j' \notin \{0, 1728\}$	$j' = 1728$	$j' \notin \{0, 1728\}$
C					
p_a	10	5	10	10	5
g	0	0	0	0	0
C'					
p_a	37	17	37	25	25
g	10	6	8	5	7

The curve is highly non-singular.

Singularities

We can find all singularities of C'. In the generic case of $j \neq j'$:

- (O, O') is an singularity with multiplicity 4.
- $\{(t, t') \in E_1[2] \setminus O \times E_2[2] \setminus O'\}$ is a set of 9 singularities with multiplicity 2. C' does not pass through other torsion points than the ones mentioned.
- $V(x = t = 0)$ is a set of 4 singularities with multiplicity 3.

This fully explains the discrepancy between arithmetic and geometric genus of C'. We can moreover explain the rationality of C by Riemann-Hurwitz.

Intersection numbers

The Néron-Severi $NS(A)$ is generated by the two axes h, v. I computed the intersection numbers $h.C' = v.C' = 6$. Via adjunction formula, this yields the expected arithmetic genus 37.

Surprise equations

It turns out (after some computation) that this is because C' is cut out by the following quadric:

$$(ct + e)y_1^2 = (ax + b)y_2^2$$

Conclusion: The Truth Is Out There

I have figured out *what* this curve is but not *why* it is there. Furthermore, I saw that the above equation still yields a rational curve when we slightly modify X, e.g. changing the powers of y or x and t.