Algorithms for the Approximate Common Divisor Problem

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Thanks

- Referees and Program Committee
- Nadia Heninger
- Tancrède Lepoint

Outline

- Approximate common divisor problem (ACD).
- Simultaneous Diophantine approximation.
- Orthogonal lattice method.
- Multivariate polynomial approach.
- Main conclusion: multivariate polynomial approach is not better than the other lattice methods for practical cryptanalysis.
- Sample-amplification and pre-processing approaches.
- Open problems.

Approximate Common Divisor problem (ACD)

- Introduced by Howgrave-Graham.
- Given $x_i = pq_i + r_i$ with $|r_i| \ll p$ for $1 \leq i \leq t$ to compute p.
- This is a well-defined problem if one is given enough samples.

Homomorphic Encryption

- Van Dijk, Gentry, Halevi and Vaikuntanathan proposed a homomorphic encryption scheme based on ACD.
- Ciphertexts are $c = pq + 2r + m$ where $m \in \{0, 1\}$ is message and $|r| \ll p$.
- To decrypt: reduce modulo p and then modulo 2.
- Homomorphic for addition:

$$c_1 + c_2 = p(q_1 + q_2) + 2(r_1 + r_2) + (m_1 + m_2)$$

  decrypts to $m_1 + m_2 \pmod 2$.
- Homomorphic for multiplication:

$$c_1 c_2 = p(pq_1 q_2 + 2q_1 r_2 + 2q_2 r_1) + 2(2r_1 r_2 + r_1 m_2 + r_2 m_1) + (m_1 m_2)$$

  which decrypts to $m_1 m_2 \pmod 2$ as long as $2r_1 r_2 \ll p$.

Further variants

- J.-S. Coron, A. Mandal, D. Naccache, M. Tibouchi. Fully homomorphic encryption over the integers with shorter public keys. CRYPTO 2011.
- J.-S. Coron, D. Naccache, M. Tibouchi. Public Key Compression and Modulus Switching for Fully Homomorphic Encryption over the Integers. EUROCRYPT 2012.
- J. H. Cheon, J.-S. Coron, J. Kim, M. S. Lee, T. Lepoint, M. Tibouchi, A. Yun. Batch Fully Homomorphic Encryption over the Integers. EUROCRYPT 2013.
- T. Lepoint, Design and Implementation of Lattice-Based Cryptography, PhD thesis 2014.
- J. H. Cheon, D. Stehlé. Fully Homomorphic Encryption over the Integers Revisited. EUROCRYPT 2015.

Cheon and Stehlé variant

- New harder variant of the problem: If LWE hard then ACD hard.
- More efficient homomorphic encryption using "scale invariant" concept.

Formal ACD problem

- Fix $\gamma, \eta, \rho \in \mathbb{N}$ with $\gamma > \eta > \rho$.
- p is an η-bit odd integer.
- Define

$$\mathcal{D}_{\gamma,\rho}(p) = \{pq + r \mid q \leftarrow \mathbb{Z} \cap [0, 2^\gamma/p), r \leftarrow \mathbb{Z} \cap (-2^\rho, 2^\rho)\}.$$

- **Approximate common divisor problem (ACD)**: Given polynomially many samples x_i from $\mathcal{D}_{\gamma,\rho}(p)$, to compute p.
- **Partial approximate common divisor problem (PACD)**: Given polynomially many samples x_i from $\mathcal{D}_{\gamma,\rho}(p)$ and also a sample $x_0 = pq_0$ for uniformly chosen $q_0 \in \mathbb{Z} \cap [0, 2^\gamma/p)$, to compute p.
- There are also "decisional" versions.

Parameters

- Let λ be a security parameter.
- Take $\rho = \lambda$ due to attacks on the term r in $pq + r$. See Chen-Nguyen, Coron-Naccache-Tibouchi, Lee-Seo.
- Van Dijk et al set $\gamma/\eta^2 = \omega(\log(\lambda))$ to thwart lattice attacks on the approximate common divisor problem.
- Suggested parameters $(\rho, \eta, \gamma) = (\lambda, \lambda^2, \lambda^5)$
- One example $(\rho, \eta, \gamma) = (71, 2698, 19350000)$.
  Yes, each ACD sample $x_i = pq_i + r_i$ is 19 million bits (about 2.4 megabytes).

Variants

- **CRT-ACD problem**
- Cheon et al set $\pi = p_1 \cdots p_\ell$ and $x_0 = \pi q_0$.
- A ciphertext is $c = \pi q + r \equiv 2r_r + m_i \pmod{p_i}$ for all i.
- Problem is to compute $p_1, \ldots, p_\ell$.
- It is an open problem to give an algorithm to solve the CRT-ACD problem that exploits the CRT structure.
- **Cheon-Stehlé approximate common divisor problem**
- Parameters

$$(\rho, \eta, \gamma) = (\lambda, \lambda + d\log(\lambda), \Omega(d^2\lambda\log(\lambda))),$$

  where d is the homomorphic circuit depth.
- Note that ρ is no longer extremely small compared with η.

Simultaneous Diophantine approximation approach (SDA)

- Due to Howgrave-Graham.
- Does not benefit from having an exact sample $x_0 = pq_0$, so suppose $x_0 = pq_0 + r_0$.
- If $x_i = pq_i + r_i$ for $1 \leq i \leq t$, where r_i is small, then

$$\frac{x_i}{x_0} \approx \frac{q_i}{q_0}$$

  for $1 \leq i \leq t$.
- In other words, the fractions q_i/q_0 are an instance of simultaneous Diophantine approximation to x_i/x_0.

Simultaneous Diophantine approximation approach (SDA)

Define lattice L of rank $t+1$ with (row) basis

$$\mathbf{B} = \begin{pmatrix} 2^{\rho+1} & x_1 & x_2 & \cdots & x_t \\ & -x_0 & & & \\ & & -x_0 & & \\ & & & \ddots & \\ & & & & -x_0 \end{pmatrix}.$$

Note $\det(L) = 2^{\rho+1} x_0^t$.
Note that L contains the vector

$$\begin{aligned} \mathbf{v} &= (q_0, q_1, \cdots, q_t)\mathbf{B} \\ &= (2^{\rho+1} q_0, q_0 x_1 - q_1 x_0, \cdots, q_0 x_t - q_t x_0) \\ &= (q_0 2^{\rho+1}, q_0 r_1 - q_1 r_0, \cdots, q_0 r_t - q_t r_0). \end{aligned}$$

SDA algorithm

- If

$$\|\mathbf{v}\| \approx \sqrt{t+1}\, 2^{\gamma-\eta+\rho+1} < \sqrt{\frac{t+1}{2\pi e}} \det(L)^{1/(t+1)}$$

  then we expect target vector $\mathbf{v}$ to be the shortest non-zero vector in the lattice.
- The attack is to run a lattice basis reduction algorithm to get a candidate $\mathbf{w}$ for the shortest non-zero vector.
- One then divides the first entry of $\mathbf{w}$ by $2^{\rho+1}$ to get a candidate value for q_0 and then computes $r_0 = x_0 \pmod{q_0}$ and $p = (x_0 - r_0)/q_0$.
- One can then "test" this value for p by checking if $x_i \pmod{p}$ are small for all $1 \leq i \leq t$.

Remarks

- Attack only requires a single short vector, not a large number of short vectors.
- Analysis of the attack is heuristic.
- To use LLL, need target $\mathbf{v}$ to be shorter by an exponential factor than the second successive minimum. So need

$$2^{t/2}\|\mathbf{v}\| \leq \sqrt{n}\det(L)^{1/(t+1)}.$$

- Necessary condition for algorithm to succeed is

$$t+1 > \frac{\gamma-\rho}{\eta-\rho}.$$

- Consistent with work of Cheon-Stehlé.
- See paper for more details and discussion.

CRT case

- Have $x_i = p_j q_{i,j} + r_{i,j}$ for $1 \leq j \leq \ell$ where each $r_{i,j}$ is small.
- It follows that the lattice contains the vectors

$$(q_{0,j} 2^{\rho+1}, q_{0,j} r_{1,j} - q_{1,j} r_{0,j}, \cdots, q_{0,j} r_{t,j} - q_{t,j} r_{0,j})$$

  for all $1 \leq j \leq \ell$ and these all have similar length.
- The j-th vector allows to compute p_j.
- But any short linear combination of several of these vectors is also a short vector in the lattice, but not good for breaking the system.

Orthogonal Lattice Approach (OL)

- Nguyen and Stern promoted the orthogonal lattice for cryptanalysis.
- Appendix B.1 of van Dijk et al gives a method based on vectors orthogonal to $(x_1, \ldots, x_t)$.
  Their idea is that the lattice of integer vectors orthogonal to $(x_1, \ldots, x_t)$ contains the sublattice of integer vectors orthogonal to both $(q_1, \ldots, q_t)$ and $(r_1, \ldots, r_t)$.
- They also have a method based on vectors orthogonal to $(1, -r_1/R, \ldots, -r_t/R)$, where $R = 2^\rho$.
- Ding and Tao have given a method based on vectors orthogonal to $(q_1, \ldots, q_t)$.
- Cheon and Stehlé considered the second method of DGHV.
- Our analysis and experiments suggest all these methods are essentially equivalent in both theory and practice.

Orthogonal Lattice Approach (OL)

- Need to have $t-1$ linearly independent vectors in the lattice L that satisfy a certain bound.
- Our approach is a bit simpler than previous works.
- We show that a necessary condition on the dimension is $t \geq (\gamma - \rho)/(\eta - \rho)$.
  Same as the SDA condition.
- In practice the OL method slightly faster than SDA as numbers smaller.

Multivariate polynomial approach (MP)

- Howgrave-Graham was the first reduce the approximate common divisor problem to the problem of finding small roots of multivariate polynomial equations.
- The idea was further extended in Appendix B.2 of van Dijk et al.
- A detailed analysis was given by Cohn and Heninger in ANTS 2012.
- A variant for the case when the "errors" are not all the same size was given by Takayasu and Kunihiro.
- Cohn and Heninger show that this approach has advantages over the others if the number of ACD samples is very small (the original context studied by Howgrave-Graham).
- Our heuristic analysis and experimental results suggest that the multivariate approach has no advantage over the SDA or OL methods for practical cryptanalysis.

Multivariate polynomial approach (MP)

PROBABLY SKIP ALL THE DETAILS

Multivariate polynomial approach (MP)

- Notation from Cohn and Heninger:
- Assume we have $N = pq_0$.
- Let $a_i = pq_i + r_i$ for $1 \leq i \leq m$ be ACD samples, where $|r_i| \leq R$ for some given bound R.
- Construct a polynomial $Q(X_1, X_2, \ldots, X_m)$ in m variables such that $Q(r_1, \cdots, r_m) \equiv 0 \mod p^k$ for some k. T
- Such polynomials are integer linear combinations of

$$(X_1 - a_1)^{i_1} \cdots (X_m - a_m)^{i_m} N^{\ell}$$

  where ℓ is chosen such that $i_1 + \cdots + i_m + \ell \geq k$.
- An additional generality is to choose a degree bound $t \geq k$ and impose the condition $i_1 + \cdots + i_m \leq t$.
- The value t will be optimised later.
- There is no benefit to taking $k > t$.

Multivariate polynomial approach (MP)

- The lattice L has dimension $d = \binom{t+m}{m}$ and determinant

$$\det(L) = R^{\binom{t+m}{m}\frac{mt}{m+1}} N^{\binom{k+m}{m}\frac{k}{m+1}} = 2^{d\frac{\rho mt}{m+1}+\binom{k+m}{m}\frac{\gamma k}{m+1}}$$

  where we use the natural choice $R = 2^{\rho}$.
- Let $\mathbf{v}$ be a vector in L.
- One can interpret $\mathbf{v} = (v_{i_1,\cdots,i_m} R^{i_1+\cdots+i_m})$ as the coefficient vector of a polynomial

$$Q(X_1, \ldots, X_m) = \sum_{i_1,\cdots,i_m} v_{i_1,\cdots,i_m} X_1^{i_1} \cdots X_m^{i_m}.$$

Multivariate polynomial approach (MP)

- So a short vector in L gives a polynomial Q.
- If $|Q(r_1, \cdots, r_m)| < p^k$ then we have $Q(r_1, \cdots, r_m) = 0$ over the integers.
  We have

$$\begin{aligned} |Q(r_1, \cdots, r_m)| &\leq \sum_{i_1, \cdots, i_m} |v_{i_1 \cdots i_m}| |r_1|^{i_1} \cdots |r_m|^{i_m} \\ &\leq \sum_{i_1, \cdots, i_m} |v_{i_1 \cdots i_m}| R^{i_1} \cdots R^{i_m} \\ &= \|\mathbf{v}\|_1. \end{aligned}$$

- Hence, if $\|\mathbf{v}\|_1 < p^k$ then we have an integer polynomial with the desired root.

Multivariate polynomial approach (MP)

- We call a vector $\mathbf{v} \in L$ such that $\|\mathbf{v}\|_1 < p^k$ a **target vector**.
- We need (t least m algebraically independent target vectors.
- Elimination leads to $(r_1, \ldots, r_m)$.
- One then computes $p = \gcd(N, a_1 - r_1)$.
- We call this process the **MP algorithm**.
- The case $(t, k) = (1, 1)$ gives the OL method, as noted by van Dijk et al.
  Cohn-Heninger call $(t, k) = (1, 1)$ "unoptimised".
- Does taking $t > 1$ gives rise to a better attack?
- When the number of ACD samples is large the best choice for MP algorithm is $(t, k) = (1, 1)$.

Multivariate polynomial approach (MP)

- Necessary condition for success using LLL is

$$d\log_2(d) + d^2\log_2(1.02) + d\rho\frac{mt}{m+1} + \gamma\binom{k+m}{m}\frac{k}{m+1} < k\eta d.$$

- This is equation (5.2) in our paper.
- Cohn-Heninger fix m, set $\beta = \eta/\gamma \ll 1$, and impose $t \approx \beta^{-1/m}k$, which means that $t \gg k$.
- The lattice dimension in their method is $\binom{t+m}{m} = O(t^m) = O(\beta^{-1}k^m) > \gamma/\eta$.
  This is the same dimension bound as previous methods (at least, when ρ is small).

Multivariate polynomial approach (MP)

- For large m, $\frac{mt}{m+1} \approx t$. To satisfy (5.2) need

$$t\rho < k\eta.$$

- Equation (5.2) implies, when m is large,

$$d\rho t \ + \ \gamma \binom{k+m}{m} \tfrac{k}{m+1} < k\eta d.$$

- Dividing by k and re-arranging gives

$$d > \frac{\gamma}{\eta - \frac{t}{k}\rho} \binom{k+m}{m} \frac{1}{m+1}.$$

Since $\frac{t}{k} \geq 1$ and $\binom{k+m}{m}\frac{1}{m+1} \geq 1$ we see that this is never better than the lattice dimension bound $d > \frac{\gamma}{\eta-\rho}$.

Executive summary

- There is no theoretical reason why, when m is large, the MP method should be better than the SDA or OL methods for any of the variants of the ACD problem.
- a special case $(t, k) = (1, 1)$ of the MP method gives the OL method. This was noted by van Dijk et al, and Cohn-Heninger call $(t, k) = (1, 1)$ "unoptimised".
- Our practical experiments confirm this, and indeed show the MP algorithm with $(t, k) \neq (1, 1)$ is very slow due to solving systems of polynomial equations.
- When m is very small then one can handle larger errors using the multivariate polynomial approach than SDA or OL (see ANTS 2012).

Pre-processing of the ACD samples

- Most important factor in the difficulty of the ACD problem is the ratio γ/η.
- If can lower γ without changing the size of the errors then have an easier instance.
- Hence, we consider a pre-processing step where a large number of initial samples $x_i = pq_i + r_i$ are used to form new samples $x'_j = pq'_j + r'_j$ with q'_j significantly smaller than q_i.
- Take differences $x_k - x_i$ for $x_k > x_i$ and $x_k \approx x_i$.
- Note that if $x_k \approx x_i$ then $q_k \approx q_i$ but r_k and r_i are not necessarily related at all.
- Hence $x_k - x_i = p(q_k - q_i) + (r_k - r_i)$ is an ACD sample for the same p, with smaller value for q and a similar sized error r.

Pre-processing of the ACD samples

- We also propose a **sample amplification** idea to convert a small list of samples into a large list, so that the method can be iterated.
- This approach looks stupid: Why not just build a lattice from all the samples.
- But the number of samples may be astronomically large.

Blum-Kalai-Wasserman (BKW) algorithm

- Our work is inspired by the BKW algorithm for learning parity with noise (LPN).
- In that case we have samples $(\mathbf{a}, b)$ where $\mathbf{a} \in \mathbb{Z}_2^n$ is a vector of length n and $b = \mathbf{a} \cdot \mathbf{s} + e$, where $\mathbf{s} \in \mathbb{Z}_2^n$ is a secret and e is a noise term which is usually zero.
- To obtain samples such that $\mathbf{a} = (1, 0, 0, \ldots, 0)$, or similar, iterate by adding samples $(\mathbf{a}_k, b_k) + (\mathbf{a}_i, b_i)$ where some coordinates of $\mathbf{a}_k$ and $\mathbf{a}_i$ agree.
- The result is an algorithm with subexponential complexity $2^{n/\log(n)}$, compared with the naive algorithm (guessing all $\mathbf{s} \in \mathbb{Z}_2^n$) which has complexity 2^n.
- In our context we do not have $(q_i, pq_i + r_i)$ but only $x_i = pq_i + r_i$, however we can use the high-order bits of x_i as a proxy for the high order bits of q_i and hence perform a similar algorithm.

Preserving the sample size

- Fix a small bound $B = 2^b$ (e.g., $B = 16$) and select B samples $x_1, \ldots, x_B$ such that the leading coefficients in base B are all distinct.
- For each of the remaining $\tau - B$ samples, generate a new sample by subtracting the one with the same leading coefficient.
- The result is $\tau - B$ samples each of size $\gamma - b = \gamma - \log_2(B)$ bits.

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- The result is $\tau - B$ samples each of size $\gamma - b = \gamma - \log_2(B)$ bits.
- Easy to see this is stupid.

Aggressive shortening

- Sort the samples $x_1 \leq x_2 \leq \cdots \leq x_\tau$ and, for some small threshold $T = 2^{\gamma-\mu}$, generate new samples by subtracting $x_{i+1} - x_i$ when this difference is less than T.
- The new samples are of size at most $\gamma - \mu$ bits, but there are far fewer of them.
- The statistical distribution of such "spacings" was considered by Pyke.
  It is shown that generic spacings have Exponential distributions.
- Eventually one has too few samples.

Sample amplification

- Generate new samples of about the same bitlength by taking sums/differences of the initial list of samples.
- Let $\mathcal{L} = \{x_1, \ldots, x_\tau\}$ be a list of ACD samples, with $x_k = pq_k + r_k$ having mean and variance given by $\mu = \mathbf{E}(x_k) = p\mathbf{E}(q_k) = 2^{\gamma-1}$ and variance given by

$$\begin{aligned} \mathrm{Var}(x_k) &= p^2\mathrm{Var}(q_k) + \mathrm{Var}(r_k) = \tfrac{1}{3}2^{2(\gamma-1)} + \tfrac{1}{12}2^{2\rho} \\ &= \tfrac{1}{3}2^{2(\gamma-1)}\left(1 + 2^{-2(\gamma-\rho)}\right). \end{aligned}$$

- Generate m random sums

$$S_k = \sum_{i=1}^{\ell} x_{k_i} \qquad [k = 1, \ldots, m],$$

which have mean and variance given by

$$\mathbf{E}(S_k) = l2^{\gamma-1} \text{ and } \mathrm{Var}(S_k) = \tfrac{1}{3}l2^{2(\gamma-1)}\left(1 + 2^{-2(\gamma-\rho)}\right).$$

Aggressive shortening

- Start with a list $\mathcal{L} = \{x_1, \ldots, x_\tau\}$ of ACD samples of mean value $2^{\gamma-1}$ and standard deviation $\sigma_0 \approx 3^{-\frac{1}{2}} 2^{(\gamma-1)}$.
- Amplify this to a list of m samples S_k.
- Sort the S_k to get the spacings $S_{k+1} - S_k$.
- Store the $\tau = m/2$ "middle" spacings as input to the next iteration of the algorithm.
- After an appropriate number of iterations run the orthogonal lattice attack.

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- Store the $\tau = m/2$ "middle" spacings as input to the next iteration of the algorithm.
- After an appropriate number of iterations run the orthogonal lattice attack.
- Conclusion: It still doesn't work, the number of iterations required is just too large.

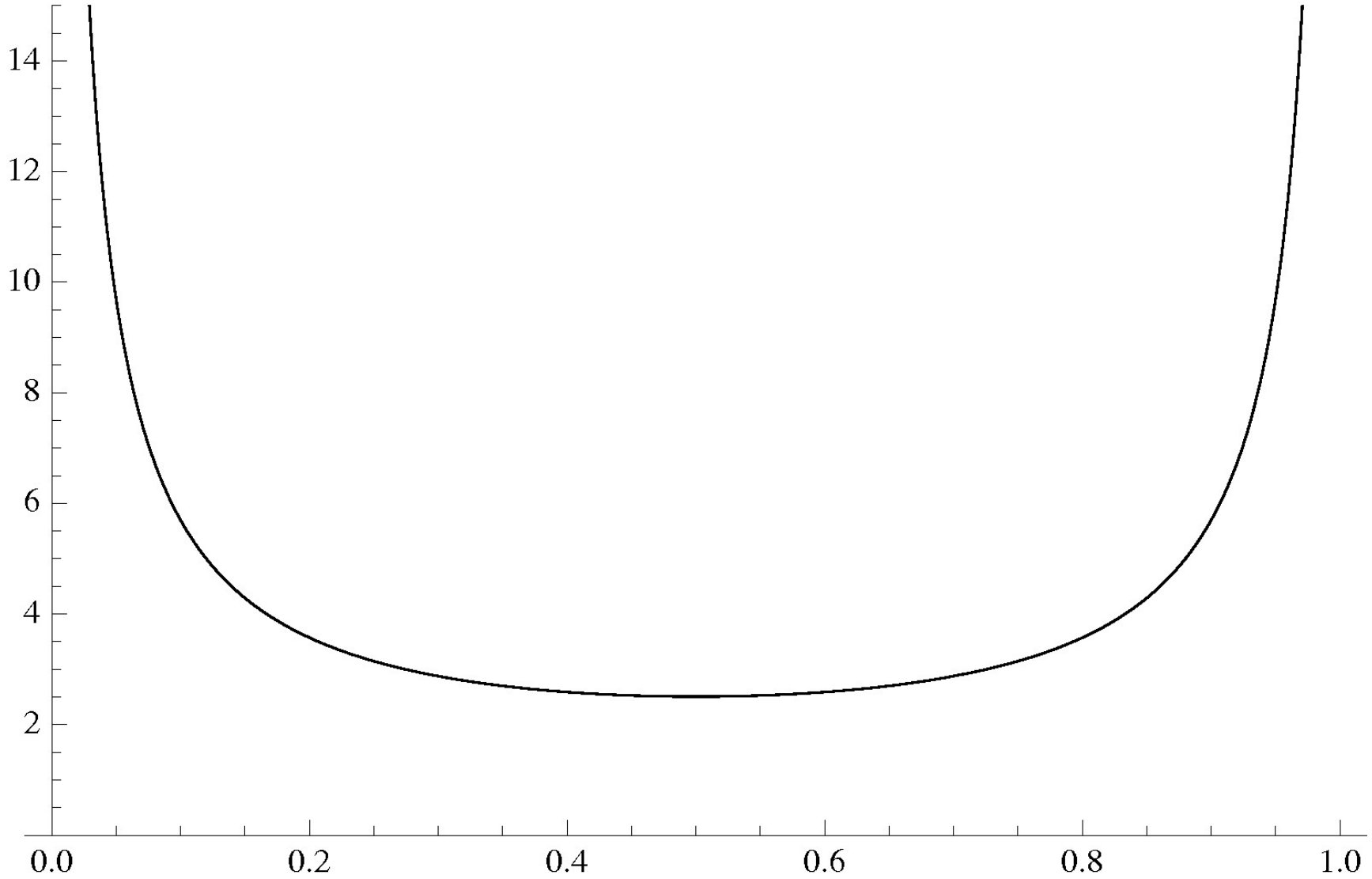
Contributions

- We obtained a refined lower bound $(\gamma - \rho)/(\eta - \rho)$ on the dimension of lattices in the SDA and OL algorithms.
- We showed that all orthogonal lattice methods for ACD are basically the same.
- We showed the multivariate polynomial method is not better than other methods for cryptanalysis of homomorphic encryption schemes based on ACD.
- We explored an analogue of the BKW algorithm for ACD and showed that it doesn't work.

Open problems

- Find improved algorithms for the CRT-ACD problem.
- Find improved algorithms for partial ACD (i.e., when one is given an exact multiple pq_0 of p).

Thank you for your attention



Thank you for your attention