Effective Hasse principle for two quadrics

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ANTS XII

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The variety V_{q_0,q_1} is *smooth* if the rank of the Jacobian of q_0 and q_1 is equal to 2 at every point of $V_{q_0,q_1}(\overline{\mathbb{Q}})$.

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We say that Q_0 and Q_1 defined over $\mathbb{Q}$, satisfy the *condition* 1 if $\det(Q_0) \neq 0$, and V_{q_0,q_1} is smooth over $\mathbb{Q}$.

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Aim

Assuming condition 1, exhibit a rational point $x \in V_{q_0,q_1}(\mathbb{Q})$ if there is one.

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Step 2

If q_0 and q_1 have a zero over $\mathbb{R}$, compute a zero over $\mathbb{Q}$.

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Theorem

The quadratic forms q_0 and q_1 have a zero over $\mathbb{R}$ if and only if every forms in $\mathcal{P}_{\mathbb{R}}(q_0, q_1)$ is indefinite.

Forms in the pencil

Definition

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Lemma

Let Q_0 and Q_1 be two matrices of size n satisfying condition 1. Assume that $\Delta(\lambda)$ has $m \leq n$ real roots denoted by $\lambda_1 < \ldots < \lambda_m$. Set $\lambda_0 := -\infty$ and $\lambda_{n+1} := +\infty$. We have:

1. *If $m \neq n$ then $\lambda Q_0 + Q_1$ is never definite.*
2. *If $m = n$, there exists at most one interval $]\lambda_i, \lambda_{i+1}[$ over which $\lambda Q_0 + Q_1$ is positive definite. Moreover this interval is $]\lambda_s, \lambda_{s+1}[$ where $[r, s]$ is the signature of Q_0.*

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5. If $\lambda Q_0 + Q_1$ is definite, return λ. If $\mu Q_0 + Q_1$ is definite, return μ.
6. Return a message saying that $\lambda Q_0 + Q_1$ is never definite.

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Proof

Assume $d(-\infty) = -a$ and $d(\infty) = a$. As Q_0 and Q_1 satisfy condition 1, d is piecewise constant. Moreover if λ_i is a real root of $\Delta(\lambda)$, we have $d(\lambda_i^-) = d(\lambda_i^+) \pm 2$. So there exists an interval I such that, for all x in I, $|d(x)| \leq 1$.

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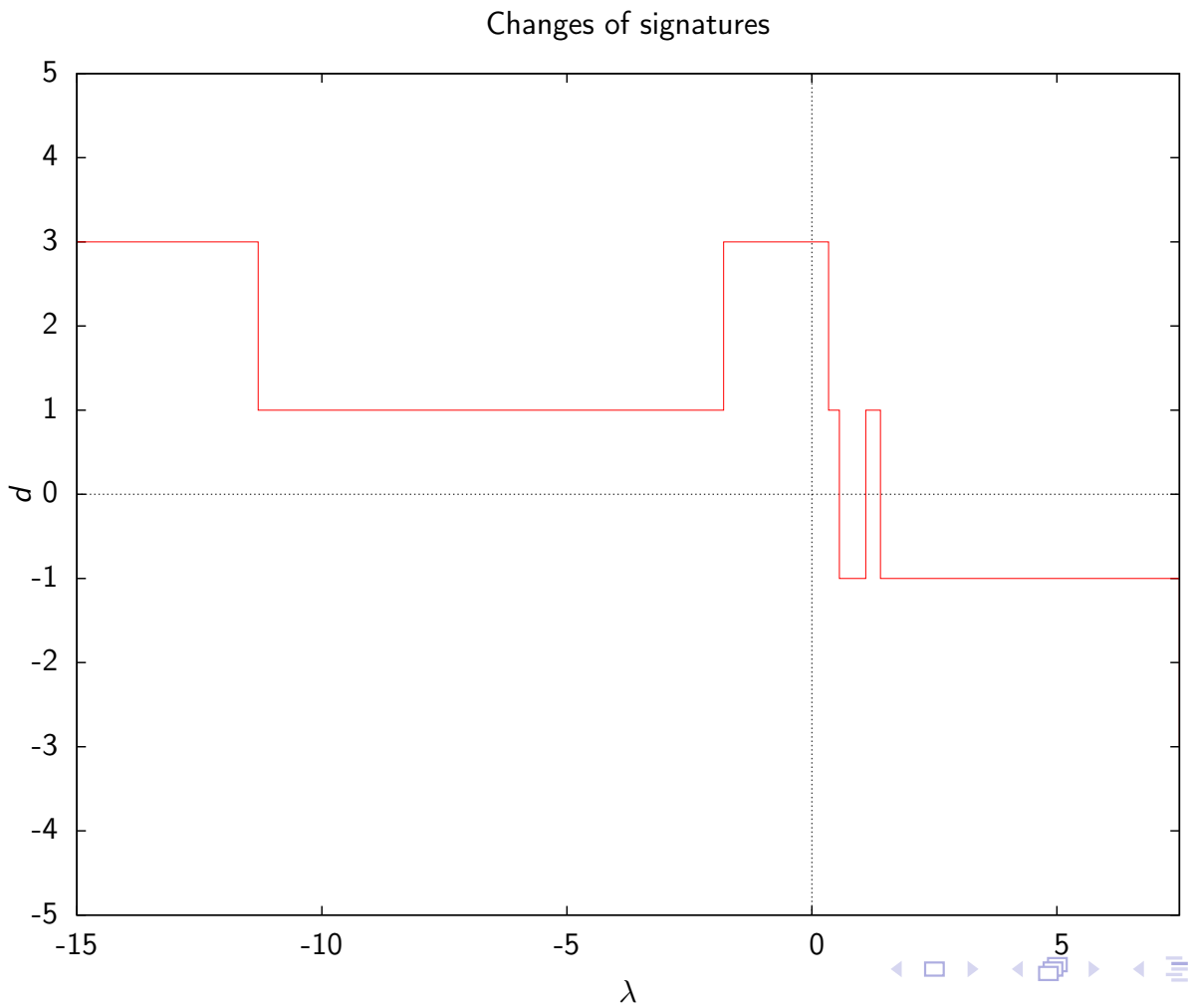

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Principle: bissection method.

In this algorithm, we don't need to compute the roots of Δ.

Reduction of q_0

We can now assume that q_0 is balanced and of signature $[7, 6]$. A rational indefinite quadratic form in $n \geq 5$ variables is always isotropic (Meyer).

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We can now assume that q_0 is balanced and of signature $[7, 6]$. A rational indefinite quadratic form in $n \geq 5$ variables is always isotropic (Meyer). Then, we can compute a rational zero of q_0 using the algorithm of Castel and a basis change such that

$$q_0 = x_1 x_{13} + q_2(x_2, \ldots, x_{12}),$$

in this basis. The signature of q_2 is $[6, 5]$, then we can iterate to have

$$q_0 = x_1 x_{13} + \ldots + x_5 x_9 + q_2(x_6, x_7, x_8).$$

MTIS of q_0

Let $W = \{x \in \mathbb{Q}^{13} \mid x_i = 0 \ \forall i > 5\}$ be a MTIS of q_0. The restriction $q_{1|_W}$ is a quadratic form in 5 variables.

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If $q_{1|_W}$ is indefinite, we compute a zero of $q_{1|_W}$ and we deduce a zero of q_0 and q_1.

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If $q_{1|_W}$ is indefinite, we compute a zero of $q_{1|_W}$ and we deduce a zero of q_0 and q_1.

Otherwise, we need to construct a MTIS W' of q_0 such that $q_{1|_{W'}}$ is indefinite.

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Rational zero of q_0 and q_1

We complete x and z in a basis of $\mathbb{Q}^{13}$ such that

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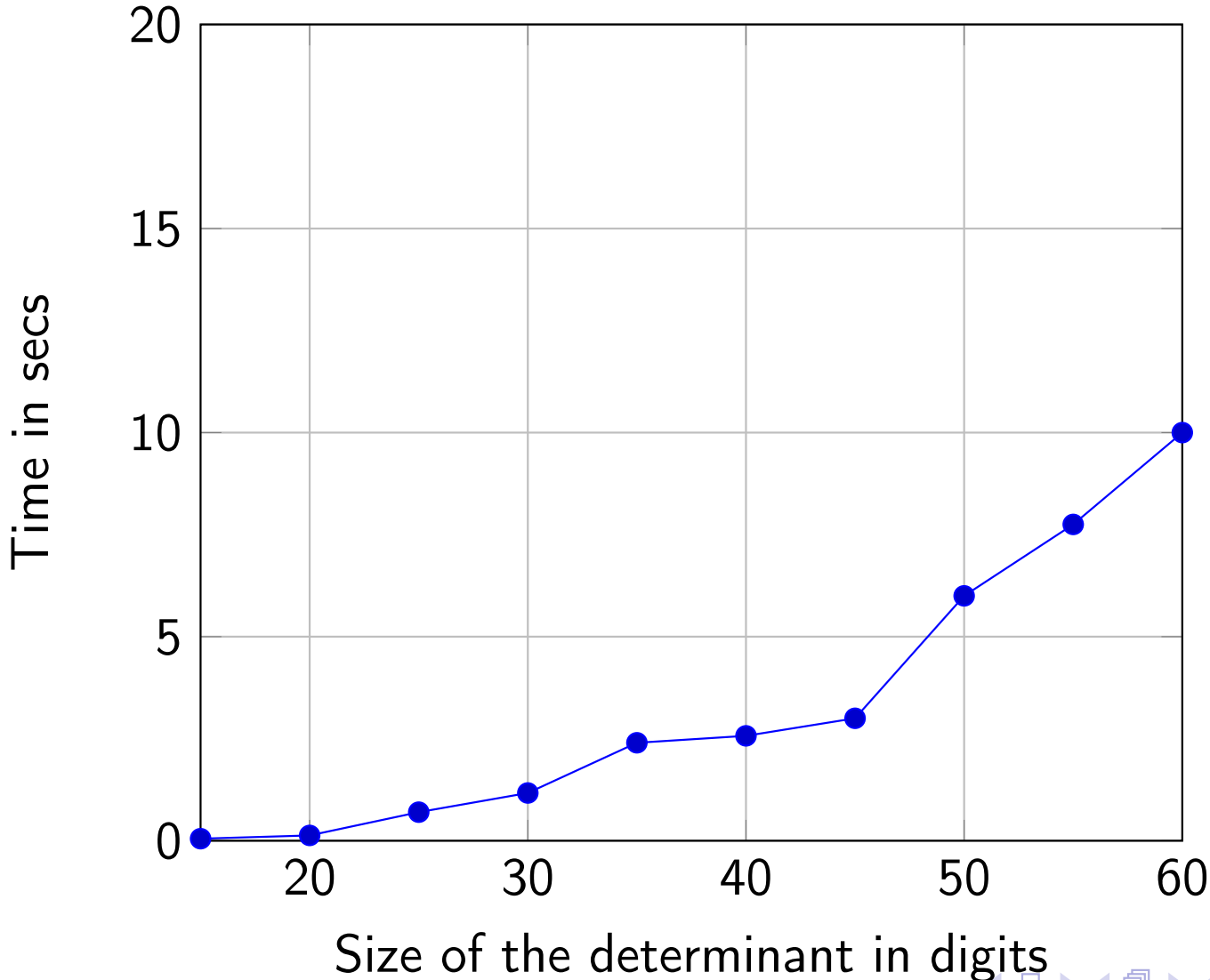
We can now compute a rational zero of $q_{1|_W}$ and deduce a zero of q_0 and q_1.

Time


20
15
10
5
0
Time in secs
20
30
40
50
60
Size of the determinant in digits

Thanks for your attention